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# Stable cut loci on surfaces

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## Introduction

Let $M$ be a 2-dimensional compact connected smooth manifold without boundary. Let $p \in M$ be fixed. Take a geodesic $g(t)$, $0 \leq t \leq \infty$, starting at $p$. Then the first point on this geodesic where the geodesic ceases to minimize distance from $p$ is called the cut point of $p$ along the geodesic $g(t)$. The cut locus $C(p)$ is the set of all cut points of $p$. Since $M$ is compact, $C(p) \neq \emptyset$. The graph $G$ is said to be smoothly embedded in $M$ if for every point $q \in G$, there exists a smooth coordinate chart $\rho : V \to R^2$ where $V$ is an open neighborhood of $q$ in $M$, such that, for every edge $e$ of $G$ with $q \in e$, $\rho(e \cap V)$ is contained in a 1-dimensional affine subspace of $R^2$. Suppose $G$ is a connected finite graph which is smoothly embedded in $M$, and whose vertices have degree 1 or 3 only. Furthermore, suppose that for every vertex $v$ of $G$ of degree 3, the tangent vectors to $M$ at $v$ in the directions of the three edges of $G$ incident to $v$ are not contained in a closed half-space of $T_vM$. Also, suppose that the inclusion map $\iota : G \to M$ induces an isomorphism $\iota_* : H_1(G; Z/2) \to H_1(M; Z/2)$. In §1 - 3, with the preceding hypothesis, we construct a smooth Riemannian metric $\alpha$ on $M$ and find a point $p \in M$ so that the cut locus $C(p, \alpha)$ of $p$ with respect to $\alpha$ is $G$, and in §4, we show that the cut locus $C(p, \alpha)$ is stable for $\alpha$.

## 1 Construction of the model curves

Let $\gamma : R \to R^2$ be a $C^\infty$ unit speed plane curve.

Then $\gamma'(t) = T_\gamma(t)$ and $T'_\gamma(t) = \kappa_\gamma(t)N_\gamma(t)$ where $T_\gamma(t)$ is the unit tangent vector of $\gamma(t)$, $N_\gamma(t)$ is the unit normal vector of $\gamma(t)$ such that $\{T_\gamma(t),\ N_\gamma(t)\}$ has the standard orientation and $\kappa_\gamma(t)$ is the signed curvature of $\gamma(t)$. The center

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of curvature of $\gamma(t)$ at $\gamma(t_0)$ is $\gamma(t_0)+\frac{N_\gamma(t_0)}{\kappa_\gamma(t_0)}$ where $\kappa_\gamma(t_0)\neq 0$. The evolute of $\gamma(t)$ is $\gamma(t)+\frac{N_\gamma(t)}{\kappa_\gamma(t)}$ where $\kappa_\gamma(t)\neq 0$. The parallel curve of $\gamma(t)$ at distance $r$ is given by $\gamma(t)+rN_\gamma(t)$. The cut point of $\gamma(t_0)$ is the first point on the normal line at $\gamma(t_0)$ in the direction of $N_\gamma(t_0)$ where the normal line ceases to minimize its distance from $\gamma$. The cut locus of $\gamma$ is the set of all cut points of $\gamma(t)$ (i. e. the cut locus of $\gamma$ is the Maxwell set of the family of parallel curves of $\gamma$ with the distance parameter). The cut point on the normal ray $\gamma(t)+uN_\gamma,\ u\geq 0$ cannot occur after the center of curvature of $\gamma(t)$. This is easy to prove. We also need the following generalization of the cut locus of the plane curve. Let $\gamma_i:[a_i,b_i]\to R^2,\ i=1,2,\dots,n$ be a finite collection of smooth disjoint arcs. For $t_0\in[a_i,b_i]$, the cut locus of $\gamma_i(t_0)$ with respect to $\gamma_1,\dots,\gamma_n$ is the first point on the normal line at $\gamma_i(t_0)$ in the direction of $N_\gamma(t_0)$ where the normal line ceases to minimize distance to the union of the arcs. The cut locus of $\{\gamma_1,\dots,\gamma_n\}$ is the set of all such cut points.

**Lemma 1.** *Let $g:[c,d]\to R$ be a $C^\infty$ function and $a,b\in R^2$ with $\|b\|=1$. There exists a unique $C^\infty$ plane curve $C$ in $R^2$ having parametrization $f$ by arc length such that if $f:[c,d]\to R^2$ then $f(c)=a,\ f'(c)=b$, and $\kappa_f(t)=g(t)$ for every $t\in[c,d]$. In other words, a plane curve is determined up to a rigid motion, by its signed curvature.*

The proof of Lemma 1 may be found in the standard Differential Geometry textbooks. Now, we are ready to construct three different types of model curves. Let $\theta$ be a variable angle such that $\frac{\pi}{2}>\theta>\frac{\pi}{3}$.

(1) Construct a curve whose curvature function is constant, i. e. an arc of a circle with angle $2\theta-\frac{2}{3}\pi$ starting from $(s_0\cos(\frac{\pi}{2}-\theta),\ l+s_0\sin(\frac{\pi}{2}-\theta))$ to $(\frac{\sqrt{3}}{2}l+s_0\cos(\theta-\frac{\pi}{6}),\ -\frac{l}{2}+s_0\sin(\theta-\frac{\pi}{6}))$, where $l$ is the given positive number and $0<s_0<\frac{\sqrt{3}l\sqrt{1+\tan(\frac{\pi}{2}-\theta)}}{2(1-\sqrt{3}\tan(\frac{\pi}{2}-\theta))}$.

(2) Construct a curve $\gamma$ satisfying the following conditions.

   a. $\kappa_\gamma(t)>0$ near $t=0,\ \kappa'_\gamma(0)=0$ and $\kappa''_\gamma(0)<0$.
   b. $\kappa_\gamma(-t)=\kappa_\gamma(t)$ and $\kappa_\gamma$ is monotonically decreasing for $t>0$.
   c. If $\gamma(t)=(X(t),Y(t))$, then $\gamma(-t)=(-X(t),Y(t)),\ \gamma(0)=(0,-\delta)$, and $X'(t),\ Y'(t)>0$ for $t>0$ where $\delta>0$.

The cut locus of $\gamma$ is contained in $Y$-axis by some consideration. Finally, we'll show that the end-point of the cut locus is an ordinary cusp of the evolute of $\gamma$. Let $F : R \times R^2 \to R$ be defined by

$$F(t,x) := (x-\gamma(t)) \cdot (x-\gamma(t)) - r^2 \text{ where } r > 0.$$

$$\frac{\partial F}{\partial t} = (x-\gamma(t)) \cdot T_\gamma(t) = 0 \text{ implies } x - \gamma(t) = \lambda N_\gamma(t) \text{ for some } \lambda.$$

Also, $F(t,x) = 0$ implies that $\lambda = \pm r$.

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2} &= -2(-T_\gamma(t) \cdot T_\gamma(t) + (x-\gamma(t)) \cdot \kappa_\gamma(t) N_\gamma(t)) \\ &= -2(-1 + (x-\gamma(t)) \cdot \kappa_\gamma(t) N_\gamma(t))\end{aligned}$$

$F = \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = 0$ implies $x = \gamma(t) + \frac{N_\gamma(t)}{\kappa_\gamma(t)}$.

Let $K : R \times R^2 \to R$ be given by

$$K(t,x) := (x-\gamma(t)) \cdot T_\gamma(t).$$

Then the discriminant set $\{\gamma(t) + \frac{N_\gamma(t)}{\kappa_\gamma(t)} | t \in (-\epsilon, \epsilon)\}$ of $K$ is the evolute of $\gamma$. (The discriminant set of $K$ is $\{x \in R^2$; there exists $t \in R$ with $F(t,x) = \frac{\partial F}{\partial t}(t,x) = 0\}$)

$$\frac{\partial K}{\partial t} = \frac{\partial^2 K}{\partial t^2} = 0 \text{ at } t = 0 \text{ if and only if}$$

$$[\kappa_\gamma(0) \neq 0,\ \kappa'_\gamma(0) = 0, \text{ and } x = \gamma(0) + \frac{N_\gamma(0)}{\kappa_\gamma(0)}],$$

since $\frac{\partial^2 K}{\partial t^2} = \frac{\kappa'_\gamma(t)}{\kappa_\gamma(t)} = 0$ at $t = 0$ (because $\kappa'_\gamma(0) = 0$)

To be an ordinary cusp of the evolute of $\gamma$, $\frac{\partial^3 K}{\partial t^3} \neq 0$ at $t = 0$. $\frac{\partial^3 K}{\partial t^3} = \frac{\kappa''_\gamma(t)}{\kappa_\gamma(t)} \neq 0$ at $t = 0$ since $\kappa''_\gamma(0) \neq 0$. (see J. W. Bruce and P. J. Giblin [2])

(3) Construct a curve $\gamma$ satisfying the following conditions:

a. $\kappa_\gamma(t) > 0$ for all $t$, $\kappa'_\gamma(0) = 0$ and $\kappa''_\gamma(0) > 0$.

b. $\kappa_\gamma(-t) = \kappa_\gamma(t)$ and $\kappa_\gamma$ is monotonically increasing for $t > 0$.

c. If $\gamma(t) = (X(t), Y(t))$, then $\gamma(-t) = (-X(t), Y(t))$, $\gamma(0) = (e, 0)$, and $X'(t),\ Y'(t) > 0$ for $t > 0$ where $e > 0$.

The initial point is $(-w + s_0 \sin\theta,\ s_0 \sin\theta)$ and initial vector is $(\cos(\frac{\pi}{2} - \theta),\ \sin(\frac{\pi}{2} - \theta))$. Also, we can construct another curve below the $X$-axis which is symmetric with respect to $X$-axis.

The parallel curves of these two curves at distance $r$ intersect each other transversely for some $r$ since the two normal lines of two curves intersect each other transversely at points of the $X$-axis between $(-w, 0)$ and $(w, 0)$ except $(0, 0)$. The cut locus of these two curves is the straight line segment from $(-w, 0)$ to $(w, 0)$.

So we have finished the local construction of three different types of model curves.

## 2 Construction of the regular neighborhoods

Let $q^1$ be a vertex of $G$ of degree 1. By definition of a smooth embedding, there exists a smooth coordinate chart $\rho : V_1 \to R^2$ where $V_1$ is an open neighborhood of $q^1$ in $M$, such that, for the unique edge $e$ of $G$ with $q^1 \in e$, $\rho(e \cap V_1)$ is contained in a ray from $\rho(q^1)$.

Let $\tau$ be a Euclidean motion (translation and rotation) of $R^2$ which takes $\rho(q^1)$ to the origin and $R$ to the positive $X$-axis. Let $\xi_1 = \tau \circ \rho$, and let $U_1 = \xi_1(V_1)$. Choose $\delta_1 > 0$ such that $B_{\delta_1}(0) = \{\, (x, y) \in R^2 \mid x^2 + y^2 < (\delta_1)^2 \,\} \subset U_1$, and $(\xi_1)^{-1}(B_{\delta_1}(0)) = V_1' \subset V_1$.

Let $q^3$ be a vertex of degree 3. By definition of a smooth embedding and our assumption on the vertices of degree 3, there exists a smooth coordinate chart $\rho : V_3 \to R^2$ where $V_3$ is an open neighborhood of $q^3$, such that $\rho(V_3 \cap G)$ is contained in three rays starting from $\rho(q^3)$ in $\rho(V_3)$ with angles which are all $< \pi$.

**Lemma 2.** *Given three rays $r_1, r_2$ and $r_3$ from $(0, 0)$ all of whose intersection angles are less than $\pi$, there exists a non-singular linear transformation $L : R^2 \to R^2$ such that $L(r_1) = \{k(1, 0) | k \geqslant 0\}$, $L(r_2) = \{k(-\frac{1}{2}, \frac{\sqrt{3}}{2}) | k \geqslant 0\}$ and $L(r_3) = \{k(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) | k \geqslant 0\}$.*

Lemma 2 is trivial since the projective group $PGL(2, R)$ acts transitively on triple of points of the projective plane.

Next, we will define a coordinate chart for each open neighborhood of a edge of $G$. Let $i = 1, 2, \ldots$, the number of vertices of degree 1 and $k = 1, 2, \ldots$, the number of vertices of degree 3. For each $i$ and $k$, we have $\delta_1^i$ and $\delta_3^k$ by the previous two constructions. Let $\delta := \min\{\delta_1^i, \delta_3^k, 1\}$. Thus we have coordinate chars $\xi_1^i \colon O_1^i \to B_\delta(0)$ and $\xi_3^k \colon O_3^k \to B_\delta(0)$ (i. e. $O_1^i = (\xi_1^i)^{-1}(B_\delta(0))$ and $O_3^k = (\xi_3^k)^{-1}(B_\delta(0))$.

Since the normal bundle of an edge $\bar{e}$ is trivial, we have a diffeomorphism $g$ from the normal bundle of $\bar{e}$ to $[-2, 2] \times R$ where the interval $[-2, 2]$ parametrizes $\bar{e}$.

By our previous construction of neighborhoods of vertices, we have coordinate charts $\tilde{\xi}_1^i\colon O_1^i \to B_\delta((2,0))$ (or $B_\delta((-2,0))$) and $\tilde{\xi}_3^k\colon O_3^k \to B_\delta((2,0))$ (or $B_\delta((-2,0))$) where the chart $\tilde{\xi}_1^i$ (resp. $\tilde{\xi}_3^k$) is obtained from the above $\xi_1^i$ (resp. $\xi_3^k$), by composition with Euclidean isometries. We choose the parametrization $[-2,2] \to \bar{e}$ so that it is equal to the inverse of the restriction of the given coordinate charts on $[-2,-2+\delta)$ and $(2-\delta,2]$. On $\bigcup_{i,k}(O_1^i \cup O_3^k)$, there is the flat metric induced by the coordinate charts. If we consider the space of metrics on $M$ as the space of sections of a fibre bundle with base $M$ and fibre the set of positive definite $(n \times n)$ matrices (see M. Buchner [3, p. 203]), we can extend this flat metric together with the metric on $\bar{e}$ induced by the given paremetrization to neighborhood of $\bar{e}$ in $M$ by the prolongation theorem for smooth sections.

Let $\exp\colon [-2,2] \times R \to M$ be the composition of $g^{-1}$ with the exponential map of the normal bundle of $\bar{e}$. Then by the tubular neighborhood theorem, exp restricts to a diffeomorphism $h$ between an open neighborhood $U_2$ of the zero section in $[-2,2] \times R$ and a neighborhood $V_2$ of $\bar{e}$. We define a new metric on $V_2$ as the flat metric induced by $h$; i. e. so that $h$ is an isometry. Since $h$ was already an isometry near the vertices of $e$, this new metric extends the flat metric defined near the vertices. Thus we obtain a flat metric on a neighborhood of $G$. Then there is a $\epsilon_0 > 0$ such that $[-2,2] \times (-\epsilon_0, \epsilon_0) \subset U_2$. Let $\epsilon' := \min\{\epsilon_0, \frac{\delta}{2}\}$ and $h^{-1}([-2.2] \times (-\epsilon', \epsilon') \subset V_2$. For any $n = 1, 2, \ldots$ the number of edges, there is $\epsilon'_n$ such that $(h_n)^{-1}([-2,2] \times (-\epsilon'_n, \epsilon'_n)) \subset V_2^n$. Let $\epsilon := \min\{\epsilon'_n\}$.

Let us define subgraphs $G_1^i$,$G_2^j$, and $G_3^k$ of the graph $G$ as follows.

(1) $G_1^i :=$ an edge $e$ together with an incident vertex of degree 1 but without incident vertex of degree 3 where $i = 1, 2, \ldots$, number of vertices of degree 1.

(2) $G_2^j :=$ an edge $e$ without two incident vertices of degree 3 where $j = 1, 2, \ldots$, number of edges with two incident vertices of degree 3.

(3) $G_3^k := G \cap O_3^k$, where $k = 1, 2, \ldots$, number of vertices of degree 3

We want to construct the neighborhoods of $G_1^i$, $G_2^j$, and $G_3^k$.

On $G_1^i$, we can get a coordinate chart $\eta_1^i : J_1^i \to \mathcal{J}_\infty^\rangle$ as follows. By our previous construction, we obtain $\mathcal{J}_\infty^\rangle = (-\epsilon, 4) \times (-\epsilon, \epsilon)$. For $p \in O_1^i$, $\eta_1^i(p) = \xi_1(p)$ and for $p \in O_2^j$, $\eta_1^i(p) = h(p) + (2,0)$ (Recall that $\delta \geq 2\epsilon > 0$). Let $J_1^i := (\eta_1^i)^{-1}(\mathcal{J}_\infty^\rangle)$.

On $G_2^j$, we just get a coordinate chart $\eta_2^j : J_2^j \to \mathcal{J}_\in^|$ by $\eta_2^j = h$, $\mathcal{J}_\in^| = (-\in, \in) \times (-\epsilon, \epsilon)$ and $J_2^j := (\eta_2^j)^{-1}(\mathcal{J}_\in^|)$.

On $G_3^k$, we get a coordinate chart $\eta_3^k : J_3^k \to \mathcal{J}_\ni^{||}$ by $\eta = \xi_3$, $\mathcal{J}_\ni^{||} = \mathcal{B}_\delta(\prime)$ and $J_3^k := (\eta_3^k)^{-1}(\mathcal{J}_\ni^{||})$.

In $J_1^i$, $J_2^j$, and $J_3^k$, we get the flat metric induced by the coordinate charts $\eta_1^i$, $\eta_2^j$, and $\eta_3^k$. Also, if $J_1^i \cap J_2^j \neq \emptyset$, $(\eta_3^k)^{-1} \circ \eta_1^i \colon J_1^i \cap J_3^k \to J_1^i \cap J_3^k$ is an isometry, and if $J_2^j \cap J_3^k \neq \emptyset$, $(\eta_3^k)^{-1} \circ (\text{Euclidean motions}) \circ \eta_2^j \colon J_2^j \cap J_3^k \to J_2^j \cap J_3^k$ is an isometry.

Let $J := \bigcup_{i,j,k}(J_1^i \cup J_2^j \cup J_3^k)$. So $J$ is a regular neighborhood of $G$ in $M$.

## 3 Construction of the metric $\alpha$

By the construction of $\mathcal{J}^{\rangle}_{\infty}$, $\mathcal{J}^{|}_{\in}$, and $\mathcal{J}^{||}_{\ni}$ in section 2, we have $\delta$ and $\epsilon$ with $\delta > 2\epsilon > 0$. Let $\delta' = \frac{3}{4}\delta$ and $s_0 = \frac{\epsilon}{\sin\theta}$.

In $\mathcal{J}^{||}_{\ni}$, draw the circle with center $(0,0)$ and radius $\delta'$ and three bands with width $\epsilon$ whose center line is the edge from $(0,0)$ to $(\delta,0)$. We get the intersection points between this circle and the boundary of the bands, and two line segments from these points to a point on the edge with intersection angle $\Theta$ with the corresponding edge. The length of each of the line segments is $s_0$. Denote the length from $(0,0)$ to a point of the edge at which the line segments intersect by $l$. By construction (I) of §1, we can get the curve $\gamma : (-c,c) \to \mathcal{J}^{\rangle}_{\infty}$ such that $\gamma(-c) = (4 - l - s_0\cos\Theta, s_0\sin\Theta)$ and $\gamma(0) = (-\frac{1}{\kappa_\gamma(0)}, 0)$.

In $\mathcal{J}^{|}_{\in}$, by construction (III) of §1, we get $\gamma : (-d,d) \to \mathcal{J}^{|}_{\in}$ such that $\gamma(-d) = (-2 + l - s_0\cos\Theta, s_0\sin\Theta)$ and $\gamma(0) = (0, A)$ for $A > 0$.

Then in $J_1^i \cap J_3^k$ or $J_2^j \cap J_3^k$, the inverse of these constructed arcs in $\mathcal{J}^{\rangle}_{\infty}$, $\mathcal{J}^{|}_{\in}$, and $\mathcal{J}^{||}_{\ni}$ under the coordinate charts fit smoothly by isometries since near their ends of the arcs have the same constant curvature.

Finally, in $J \subset M$, the union of the inverse of the constructed arcs under the coordinate charts are smooth curves by above reason. Let these curves be called $\Gamma$ and $J'$ the closed region bounded by $\Gamma$ (including $G$). Obviously $G \subset \mathrm{int}(J') \subset J' \subset J$. Consider the Mayer-Vietoris exact sequence of the pair $(J, \overline{M \setminus J'})$ with coefficients $Z/2$.

$$\begin{aligned} \cdots \to H_2(J \setminus \mathrm{int}(J')) &\to H_2(J) \oplus H_2(\overline{M \setminus J'}) \to H_2(M) \\ &\to H_1(J \setminus \mathrm{int}(J')) \to H_1(J) \oplus H_1(\overline{M \setminus J'}) \to H_1(M) \\ &\to \tilde{H}_0(J \setminus \mathrm{int}(J')) \to \tilde{H}_0(J) \oplus \tilde{H}_0(\overline{M \setminus J'}) \to \tilde{H}_0(M) \to 0 \end{aligned} \tag{1}$$

Since $M$ is a 2-dimensional compact connected manifold without boundary, $H_2(M) = Z/2$ and $\tilde{H}_0(M) = 0$. Since $\Gamma$ is a deformation retract of $J \setminus \mathrm{int}(J')$, $H_2(J \setminus \mathrm{int}(J') = 0$. Since $\overline{M \setminus J'}$ is a surface with non-empty boundary and $\overline{M \setminus J'} \subset M$, $H_2(\overline{M \setminus J'}) = 0$. Since $G$ is a deformation retract of $J'$ and $J'$ is homotopy equivalent to $J$, $H_2(J) = 0$ and $\tilde{H}_0(J) = 0$. Also, since the inclusion map $\iota : G \to M$ induces an isomorphism $\iota_* : H_1(G; Z/2) \to H_1(M; Z/2)$,

$H_1(J) = H_1(M) = (Z/2)^m$ for some positive integer $m$. Since $J \setminus \mathrm{int}(J')$ is homotopy equivalent to $\Gamma$, the inclusion maps $J \setminus \mathrm{int}(J') \to J$ and $J \setminus \mathrm{int}(J') \to \overline{M \setminus J'}$ induce the zero homomorphisms. Thus $H_1(J \setminus \mathrm{int}(J')) = H_2(M) = Z/2$. Then $\Gamma$ is connected, so $\tilde{H}_0(J \setminus \mathrm{int}(J')) = 0$. Now exactness of the Mayer-Vietoris sequence implies that $H_1(\overline{M \setminus J'}) = 0$. Also, $\tilde{H}_0(\overline{M \setminus J'}) = 0$. Thus $\overline{M \setminus J'}$ is diffeomorphic to a disk $D^2$ with $\Gamma$ mapping to $\partial D^2$.

**Proposition 1.** *Let $D$ be an $n$-disk embedded in $C^\infty$ manifold $M$. For any Riemannian metric on $M \setminus int(D)$, there is a Riemannian metric on $M$ which agrees with the original metric on $M \setminus int(D)$ such that for some $p$ in $D$, $\exp_p$ is a diffeomorphism of unit disk about the origin in $T_p(M)$ onto $M$.*

The proof of Proposition 3 can be found in Weinstein [8, Proposition C]. By Proposition 3, we can extend the flat metric constructed on the neighborhood $J'$ of the graph $G$ to a metric $\alpha$ on $M$ such that for some $p \in M \setminus J'$, $\exp_p : T_p(M) \to M$ is a diffeomorphism from the unit disk in $T_p(M)$ onto $\overline{M \setminus J'}$. In particular, the image of the unit circle in $T_p(M)$ is the curve $\Gamma$, and the geodesic rays from $p$ are orthogonal to $\Gamma$. Thus, in the isometric coordinate neighborhoods $\mathcal{J}^{\rangle}_{\infty}$, $\mathcal{J}^{|}_{\in}$, and $\mathcal{J}^{\|}_{\ni}$ constructed above, the geodesic rays from $p$ are the normal lines to the model curves. Since the metric in $J'$ is flat, the graph $G$ is the cut locus $C(p, \alpha)$.

## 4 Stability of $C(p, \alpha)$

In section 3, we have constructed a metric $\alpha$ such that $C(p, \alpha) = G$ for some $p \in M$. The cut locus $C(p, \alpha)$ is said to be stable for $\alpha$ if there is a neighborhood $W$ of $\alpha$ in the space of all metrics on $M$ with the Whitney $C^\infty$-topology such that for each $\beta \in W$, there is a diffeomorphism $A(\beta) : M \to M$ with the property that $A(\beta)(C(p, \alpha) = C(p, \beta)$ (see M. Buchner [3, 4]). In this section, we want to prove that $C(p, \alpha)$ is stable for $\alpha$. To do this, we use Looijenga's set-up (see Looijenga [5]). Let $\Gamma$, $\alpha$, and $G \subset J' \subset J$ be as in the previous section, and let $r_0$ be the largest distance from a point of $G$ to the curve $\Gamma$. From now on, we denote $\Gamma$ as $\gamma : R \to M$ like a function. Let $\delta > 0$ and $U := \{(t, x, r) : x \in J', d_\alpha(x, \gamma(t)) < r_0 + \delta, -\delta < r < r_0 + \delta\}$ where $d_\alpha$ is the distance function on $M$ corresponding to the metric $\alpha$. The family $F : U \to R$ is defined by

$$F(t, x, r) := (d_\alpha(x, \gamma(t)))^2 - r^2$$

The deformation $H : U \to R \times J' \times (-\delta, r_0 + \delta)$ associated to the family $F$ is defined by

$$H(t, x, r) := (F(t, x, r), x, r)$$

The deformation $H$ is stable if for $F'$ close to $F$ (in the Whitney $C^\infty$ topology) there exist diffeomorphisms $h$, $h'$, $h''$ such that the following diagram commutes

$$\begin{array}{ccccc} U & \xrightarrow{H} & R \times J' \times (-\delta, r_0+\delta) & \xrightarrow{\text{Proj}} & J' \times (-\delta, r_0+\delta) \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle h'}\big\downarrow & & {\scriptstyle h''}\big\downarrow \\ U & \xrightarrow{H'} & R \times J' \times (-\delta, r_0+\delta) & \xrightarrow{\text{Proj}} & J' \times (-\delta, r_0+\delta) \end{array}$$

where $H'(t,x,r) = (F'(t,x,r),x,r)$ and $h'(t,x,r) = (t, h''(x,r))$ (see Looijenga [5]). Note that $h''$ is close to the identity map and the restriction of $h$ to $J' \times (-\delta, r_0+\delta)$ is also close to the identity map.

We consider the action of $Diff(R)$ on $C^\infty(R)$ by $h \cdot f := f \circ h^{-1}$ where $Diff(R)$ is the group of diffeomorphisms from $R$ to $R$. Let $\Psi : J' \times (-\delta, r_0+\delta) \to C^\infty(R)$ be defined by

$$(\Psi(x,r))(t) := (d_\alpha(x,\gamma(t)))^2 - r^2 (= F(t,x,r))$$

By the work of Thom, Mather, and Sergeraert, the deformation $H$ is stable if and only if $\Psi$ is transverse to all the $Diff(R)$-orbits in $C^\infty(R)$ (see Looijenga [5]).

The discriminant of the family $F$ is the set

$$\begin{aligned} \mathcal{D}_F := \{(x,r) : \text{there exist} t \in R \text{such that} F(t,x,r) = 0, \\ \frac{\partial F}{\partial t}(t,x,r) = 0\} \\ = \{(x,r) : \text{there exist} t \in R \text{such that} \Psi(x,r)(t) = 0, \\ \frac{d}{dt}(\Psi(x,r)(t)) = 0\} \end{aligned}$$

First, we show that if $H$ is stable then $C(p,\alpha)$ is stable for $\alpha$. Suppose that the deformation $H$ is stable. The cut locus $G = C(p,\alpha)$ is the image by the projection $J' \times (-\delta, r_0+\delta) \to J'$ of the closure $\overline{G}$ of the double point curve of $\mathcal{D}_{\mathcal{F}}$. (To see this, consider $\mathcal{E}_F = \{(t,x,r) : F(t,x,r) = 0, \frac{\partial F}{\partial t}(t,x,r) = 0\}$, then $\mathcal{E}_F$ is a smooth surface and the double point curve of $\mathcal{D}_{\mathcal{F}}$ means the double points of the projection of $\mathcal{E}_{\mathcal{F}}$ to $(x,r)$-space.) Now, if we perturb the metric of $M$, we obtain a new cut locus $G'$, which is the projection of the cut locus $\overline{G'}$ of $\mathcal{D}_{\mathcal{F}'}$ corresponding to the perturbed family $H'(t,x,r) = (F'(t,x,r),x,r)$ where $F'$ is the corresponding perturbed family with respect to metrics. Since $H$ is stable, there exist diffeomorphisms $h, h'$ and $h''$ such that $H'(t,x,r) = (h' \circ H \circ h^{-1})(t,x,r)$ and $\overline{G'} = h''(\overline{G})$. Thus $\overline{G'} = h''(\overline{G})$ where $h''$ is a diffeomorphism.

Since the projection $J' \times (-\delta, r_0+\delta) \to J'$ restricts to a mapping $\overline{G} \to G$ and $h''$ is close to the identity, $h''$ induces a homeomorphism $\phi : G \to G'$ which extends to a diffeomorphism $\Phi : M \to M$. Thus the cut locus $G$ of $\gamma$ is stable with respect to perturbation of the metric $\alpha$ of $M$.

Next, we prove that $\Psi$ is transverse to all $Diff$-orbits on $C^\infty(R)$. Before the proof, we need to know the $Diff(R)$-orbits in $C^\infty(R)$. The orbits of singularities in $C^\infty(R)$ are followings:

(1) $f$ has one critical point $t_0$ such that $f(t_0) = f'(t_0) = 0$ and $f''(t_0) \neq 0$.

(2) $f$ has one critical point $t_0$ such that $f(t_0) = f'(t_0) = f''(t_0) = 0$ and $f'''(t_0) \neq 0$.

(3) $f$ has one critical point $t_0$ such that $f(t_0) = f'(t_0) = f''(t_0) = f'''(t_0) = 0$ and $f^{(iv)}(t_0) \neq 0$.

(4) $f$ has two distinct critical points $t_0$ and $t_1$ such that $f(t_0) = f(t_1) = f'(t_0) = f'(t_1) = 0$, $f''(t_0) \neq 0$ and $f''(t_1) \neq 0$.

(5) $f$ has three distinct critical points $t_0$, $t_1$, and $t_2$ such that $f(t_0) = f(t_1) = f(t_2) = f'(t_0) = f'(t_1) = f'(t_2) = 0$, $f''(t_0) \neq 0$, $f''(t_1) \neq 0$, and $f''(t_2) \neq 0$.

The preimage of these orbits are the strata of the discriminant locus $\mathcal{D}_{\mathcal{F}} \subset \mathcal{J}' \times (-\delta, \nabla_{\prime} + \delta)$. Type(1) are smooth points of $\mathcal{D}_{\mathcal{F}}$, type(2) are cusp points of $\mathcal{D}_{\mathcal{F}}$, type(3) are swallowtail points of $\mathcal{D}_{\mathcal{F}}$, type(4) are double points of $\mathcal{D}_{\mathcal{F}}$, type(5) are triple points of $\mathcal{D}_{\mathcal{F}}$.

To prove that $\Psi$ is transverse to all $Diff(R)$-orbits in $C^\infty(R)$, we use Mather's infinitesimal versality criterion, which we now describe (see [6]).

Consider an orbit $X \subset C^\infty(R)$ along which the function $f$ has exactly $s$ critical points $t_0, \ldots, t_{s-1}$ such that $f(t_0) = \cdots = f(t_{s-1}) = 0$. Suppose that $\Psi(x_0, r_0) = f \in X$. To prove that $\Psi$ is transverse to $X$ at $f$, we just consider the germs of $F(t, x, r)$ at $(t_i, x_0, r_0), i = 0, \ldots, s-1$. For each $i$, let $f_i(t) \in R[[t]]$ be the Taylor series of $f$ at $t_i$ (so $f_i^{(n)}(0) = f^{(n)}(t_i)$). Let $< f_i'(t) >$ be the ideal of $R[[t]]$ generated by $f_i'(t)$, and consider the $R$-algebra

$$A = \frac{R[[t]]}{< f_0'(t) >} \times \cdots \times \frac{R[[t]]}{< f_{s-1}'(t) >}.$$

Choose local coordinates $x = (u, v)$ on $M$ near $x_0$ and let $F_u$, $F_v$, $F_r$ be the elements corresponding to the functions $\frac{\partial F}{\partial u}(t, u_0, v_0, r_0)$, $\frac{\partial F}{\partial v}(t, u_0, v_0, r_0)$, $\frac{\partial F}{\partial r}(t, u_0, v_0, r_0)$, respectively. Then $\Psi$ is transverse to $X$ at $f$ if and only if $F_u$, $F_v$, $F_r$ span $A$ as a real vector space (Mather's infinitesimal criterion).

First, we consider the orbits of type(1)-(3) for which $s = 1$. Mather's criterion is easily checked for type(1). Furthermore, if $\Psi(t, x_0, r_0)$ has type(3) and $\Psi$ is transverse to the orbits of type(3) at $(x_0, r_0)$, then $\Psi$ is transverse to the orbit of type(2) for $(x, r)$ sufficiently close to $(x_0, r_0)$. The proof that $C(p, \alpha)$ is stable if $H$ is stable shows that we can replace $J'$ by an arbitrary neighborhood of the cut locus $G = C(p, \alpha)$. Thus we need only check that $\Psi$ is transverse to the orbit of type(3).

To check Mather's criterion for singularity of type(3), we can work in a coordinate patch $J_1'$ as constructed above, In these coordinates,

$$F(t, u, v, r) = ((u, v) - \gamma(t)) \cdot ((u, v) - \gamma(t)) - r^2$$

where $\gamma$ is the model curve type (II)(in §1). By this construction, $\gamma(0) = (0, 0)$ and $\gamma'(0) = (1, 0)$. Thus we can parametrize $\gamma$ as $(t, a_2t^2 + a_3t^3 + a_4t^4 + \cdots)$. Now, $F$ has a singularity at $(0, 0, \frac{1}{\kappa_\gamma(0)}, \frac{1}{\kappa_\gamma(0)})$. Let $f(t) = F(t, 0, \frac{1}{\kappa_\gamma(0)}, \frac{1}{\kappa_\gamma(0)})$. By Mather's criterion, $\Psi$ is transverse to the orbit of $f$ in $C^\infty(R)$ if and only if $\frac{\partial F}{\partial u}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))}$ generate $\frac{R[[t]]}{<f'(t)>}$ where $R[[t]]$ is the ring of power series at 0.

$$\begin{aligned} F(t, u, v, r) &= ((u, v) - \gamma(t)) \cdot ((u, v) - \gamma(t)) - r^2 \\ &= u^2 - 2ut + t^2 + v^2 - 2(a_2t^2 + a_3t^3 + a_4t^4 + \cdots)v \\ &\quad + (a_2t^2 + a_3t^3 + a_4t^4 + \cdots)^2 - r^2 \end{aligned}$$

By the construction(II) of §1, $\kappa_\gamma(0) > 0$, $\kappa_\gamma'(0) = 0$, and $\kappa_\gamma''(0) \neq 0$.

$$\begin{aligned} \frac{\partial F}{\partial u}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))} &= -2t \\ \frac{\partial F}{\partial v}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))} &= \frac{2}{\kappa_\gamma(0)} - 2(a_2t^2 + a_3t^3 + a_4t^4 + \cdots) \\ \frac{\partial F}{\partial r}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))} &= -2r = -\frac{2}{\kappa_\gamma(0)} \neq 0 \end{aligned}$$

Also, $a_2 = \frac{\kappa_\gamma(0)}{2}$, $a_3 = 0$ and $a_4 = \frac{\kappa_\gamma^3(0)}{8}$ by basic calculation. Thus

$$\begin{aligned} f(t) &= F(t, 0, \frac{1}{\kappa_\gamma(0)}, \frac{1}{\kappa_\gamma(0)}) \\ &= t^2 - \frac{2}{\kappa_\gamma(0)} \cdot (\frac{\kappa_\gamma(0)}{2}t^2 + \frac{\kappa_\gamma(0)}{8}t^4 + \cdots) \\ &\quad + (\frac{\kappa_\gamma(0)}{2}t^2 + \frac{\kappa_\gamma(0)}{8}t^4 + \cdots)^2 + \cdots \\ &= \frac{\kappa_\gamma(0)}{4}t^4 + \cdots \\ f'(t) &= \kappa_\gamma^2(0)t^3 + \cdots \end{aligned}$$

It is true that $\{1, t, t^2\}$ spans $\frac{R[[t]]}{<t^3>}$. Thus $\Psi$ is transverse to the orbit of $f$ in $C^\infty(R)$.

Next, we consider the orbit of type(4), for which $s = 2$. Now, $F$ has a type(4) singularity at $(t_0, u_0, v_0, r_0)$ and $(t_1, u_0, v_0, r_0)$ where $r_0 \neq 1/\kappa_\gamma(t_0)$, $r_0 \neq 1/\kappa_\gamma(t_1)$, $\kappa_\gamma(t_0) \neq 0$, and $\kappa_\gamma(t_1) \neq 0$. Let $g_0(t) = f(t_0 + t) = F(t_0 + t, u_0, v_0, r_0)$ and $g_1 = f(t_1 + t) = F(t_1 + t, u_0, v_0, r_0)$. By Mather's criterion, $\Psi$ is transverse to the orbits of $f$ in $C^\infty(R)$ if and only if $\frac{\partial F}{\partial u}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))}$, $\frac{\partial F}{\partial v}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))}$, $\frac{\partial F}{\partial r}|_{(u,v,r)=(0,1/\kappa_\gamma(0),1/\kappa_\gamma(0))}$ generate $\frac{R[[t]]}{<g_0'(t)>} \times \frac{R[[t]]}{<g_1'(t)>}$. Let

$$g_i(t) = f(t + t_i) = ((u_0, v_0) - \gamma(t + t_i)) \cdot ((u_0, v_0) - \gamma(t + t_i)) - r_0^2 \quad \text{for } i = 0, 1.$$

Thus $g_0(0) = f(t_0) = 0$ and $g_0'(0) = f'(t_0) = 0$ since $(u_0, v_0)$ is on the normal line at $\gamma(t_0)$. $g_0''(0) = f''(t_0) \neq 0$ since $\kappa_\gamma(t_0) \neq 0$ and $r_0 \neq \frac{1}{\kappa_\gamma(t_0)}$. Also, similarly $g_1(0) = f(t_1) = 0$, $g_1'(0) = f'(t_1) = 0$ and $g_1''(0) = f''(t_1) \neq 0$. So we have $g_0(t) = b_2 t^2 + \cdots (b_2 \neq 0)$ and $g_1(t) = c_2 t^2 + \cdots (c_2 \neq 0)$. Thus $< g_0'(t) >=< g_1'(t) >=< t >$. $\dim(\frac{R[[t]]}{<g_0'(t)>}) = \dim(\frac{R[[t]]}{<g_1'(t)>}) = 1$. It is enough to show that

$$(\frac{\partial F}{\partial u}|_{(t_0,u_0,v_0,r_0)}, \frac{\partial F}{\partial u}|_{(t_1,u_0,v_0,r_0)}) = (A, 0)$$
$$(\frac{\partial F}{\partial v}|_{(t_0,u_0,v_0,r_0)}, \frac{\partial F}{\partial v}|_{(t_1,u_0,v_0,r_0)}) = (0, B)$$

for non-zero constant $A$, $B$ since

$$(\frac{\partial F}{\partial r}|_{(t_0,u_0,v_0,r_0)}, \frac{\partial F}{\partial r}|_{(t_1,u_0,v_0,r_0)}) = (-2r_0, -2r_0).$$

To do this, we change $(u, v)$-coordinate to $(\tau, \eta)$-coordinate where the level curves of $\tau$ are the parallel curves of $\gamma$ and the level curves of $\eta$ are the parallel curves of $\gamma$ near $\gamma(t_0)$. Also, we assume that $(\tau, \eta) = (0, 0)$ at $(u, v) = (u_0, v_0)$. Thus

$$\frac{\partial F}{\partial \tau}|_{(t_0,0,0,r_0)} \neq 0 \quad \text{and} \quad \frac{\partial F}{\partial \eta}|_{(t_0,0,0,r_0)} = 0$$

since the arc of the circle through $(u_0, v_0)$ with center $\gamma(t_0)$ is tangent to the $\eta$-axis and is transverse to the $\tau$-axis. Similarly,

$$\frac{\partial F}{\partial \tau}|_{(t_1,0,0,r_0)} = 0 \quad \text{and} \quad \frac{\partial F}{\partial \eta}|_{(t_1,0,0,r_0)} \neq 0$$

and Mather's criterion holds. Also, we have another case of type(4) singularity at $(t_0, u_0, v_0, r_0)$ and $(t_1, u_0, v_0, r_0)$ where $r_0 \neq 1/\kappa_\gamma(t_0)$, $r_0 \neq 1/\kappa_\gamma(t_1)$, $\kappa_\gamma(t_0) \neq 0$ and $\kappa_\gamma(t_1) \neq 0$. To check Mather's criterion with same set-up above, we change $(u, v)$-coordinate to $(\tau, \eta)$-coordinate where the curve $\tau = 0$ is tangent to the parallel curves of $\gamma$ at $(u_0, v_0)$ and the curve $\eta = 0$ is orthogonal to the curve $\tau = 0$ at $(u_0, v_0)$. We can assume $(\tau, \eta) = (0, 0)$ at $(u, v) = (u_0, v_0)$. Then

$$\frac{\partial F}{\partial \tau}|_{(t_0,0,0,r_0)} \neq 0, \frac{\partial F}{\partial \eta}|_{(t_0,0,0,r_0)} = 0$$
$$\frac{\partial F}{\partial \tau}|_{(t_1,0,0,r_0)} \neq 0, \frac{\partial F}{\partial \eta}|_{(t_1,0,0,r_0)} = 0$$
$$\frac{\partial F}{\partial r}|_{(t_0,0,0,r_0)} \neq 0, \frac{\partial F}{\partial r}|_{(t_1,0,0,r_0)} \neq 0$$

since the arcs of the circles through $(u_0, v_0)$ with center $\gamma(t_0)$ and $\gamma(t_1)$ are tangential to the $\eta$-axis and are transverse to the $\tau$-axis. Thus Mather's criterion holds. Similarly, we can check the orbits of type(5) for which $s = 3$. Thus $\Psi$ is transverse to all $Diff(R)$-orbits in $C^\infty(R)$. This implies that $H$ is stable, so the cut locus $C(p, \alpha)$ is stable for $\alpha$. Thus we have the following theorem.

**Theorem 1.** *Let $M$ be a compact connected 2-dimensional $C^\infty$-manifold without boundary. Suppose that $G$ is a connected finite graph which is smoothly embedded in $M$, and whose vertices have degree 1 or 3 only. Furthermore, suppose that for every vertex $v$ of $G$ of degree 3, the tangent vectors to $M$ at $v$ in the directions of the three edges of $G$ incident to $v$ are not contained in a closed half-space of $T_vM$. Also, suppose that the inclusion map $\iota : G \to M$ induces an isomorphism $\iota_* : H_1(G; Z/2) \to H_1(M; Z/2)$. Then there exist a smooth metric $\alpha$ on $M$ and a point $p \in M$ so that $G = C(p, \alpha)$ and the cut locus $C(p, \alpha)$ is stable for $\alpha$.*

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*NOTE: proofs not corrected by the author.*